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Introduction

The theory and numerical treatment of inverse problems form an important area of applied mathematics. To clarify the notion of an inverse problem, we follow Keller [3] and call a pair of problems *inverse to each other*, if the formulation (i.e., the data) of each requires parts of the solution of the other problem. In this Special Issue we will concentrate on pairs of problems which involve boundary value problems for partial differential equations. Here, however, we observe a fundamental difference between each of the pairs. One of them is always *well-posed*, i.e., there *exists* a *unique* solution which *depends continuously* on the data. On the other hand, the other one is not well-posed, usually in the sense that the continuous dependence property fails to hold. For a long time, problems which were not well-posed were essentially ignored by the mathematical community. They were classed under the prejudiced titles of *ill-posed* or *improperly-posed* problems. In most cases one calls the well-posed part the *direct problem* and its ill-posed counterpart the *inverse problem*. Famous classical examples of inverse problems are the heat equation "backwards in time", parameter identification problems and inverse scattering problems. While the mathematical theory of *linear* inverse problems is relatively well understood (cf. the monographs [1,4,5]), for nonlinear inverse problems one is far away from a satisfactory unifying mathematical theory.

This Special Issue contains ten papers dealing with the theoretical and numerical aspects of *inverse scattering problems* for *acoustic* or *electromagnetic* waves. Such problems are basically both nonlinear and improperly posed. In the *direct* acoustic scattering problem, one determines the perturbation (the "scattered" field) which occurs when a known "incident" field, e.g., a plane wave, hits an obstacle of known shape and composition. Assuming periodic dependence on time ("time-harmonic case"), this leads to a boundary value problem for the Helmholtz equation in the (unbounded) region exterior to the obstacle. In the *inverse* problem, one tries to determine the shape or constitutive parameters from information about the incident and scattered fields. Important applications are ultrasound tomography and the nondestructive testing of materials.

The first two papers in this issue present numerical algorithms for determining the unknown index of refraction of a two-dimensional object. In their paper, David Colton and Peter Monk present two numerical methods which are especially successful when the medium is probed by a high number of incident waves. Both methods first determine a superposition of the incident fields such that the scattered field becomes simple and then determine the index of refraction by a nonlinear optimization process involving the Lipmann–Schwinger integral equation. In the paper by Ralph Kleinman and Peter van den Berg, a different method is proposed based on the iterative solution of the Lipmann–Schwinger equation by an over-relaxation technique. In this

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procedure, a sequence of simple minimization problems for two relaxation parameters has to be solved. Both papers contain a number of numerical results.

Alberto Grünbaum studies a potential scattering problem for the Helmholtz equation. For fixed frequency, this is formally equivalent to scattering by an inhomogeneous medium. A frequently used method for solving the (direct or inverse) problem approximately is the *Born approximation* which is the first iteration in the Neumann series solution of the Lipmann–Schwinger equation. This approximation conserves some properties of the exact solution. Alberto Grünbaum, however, shows by constructing special potentials that the classical Heisenberg uncertaincy principle, which is valid for the Born approximation, does not hold for the exact solution of the scattering problem.

The next set of three papers is concerned with the inverse obstacle problem for the Helmholtz equation. Here, the shape of a "perfectly soft" obstacle has to be determined from the scattered field far away from the obstacle. The numerical method proposed by Rainer Kress and Axel Zinn relies on analytic continuation from the far-field to the near-field by means of a single layer potential. The convergence properties of this method were discussed in earlier papers. The main step in the proof of convergence is the continuous dependence of the scattering amplitude on the geometry of the scatterer. Here, a new and more elegant proof of this result is given. The paper includes a few numerical reconstructions of three-dimensional obstacles which are not rotationally symmetric. Theodore Apostolopoulos and George Dassios propose a method based on the low-frequency expansion of the scattered field. They derive a finite set of moment equations, which suffices to solve completely the inverse scattering problem when it is a priori known that the scatterer has a polynomial boundary. Their algorithm is especially suitable for multiprocessing computers. The second part of the paper carefully analyses this algorithm with respect to efficiency and dependence on parameters. The above inverse obstacle problems are, as are all the other inverse problems of this issue, improperly posed and need some kind of stabilization. All of the well-known regularization techniques need some type of a priori information about the scatterer. Victor Isakov shows in his paper that the shape of the scatterer depends continuously on the far-field data, provided some smoothness and a priori bounds concerning the shape are known. His result is formulated in the form of a stability estimate.

In the paper by Jet Wimp, Chris Rorres and Richard Wayland Jr, analytic solutions for a family of nonuniform one-dimensional media are determined. Although this paper does not deal with the inverse problem itself, these examples could be especially useful for testing certain inverse algorithms.

The last three papers are concerned with electromagnetic inverse problems. Electromagnetic wave propagation is described by Maxwell's equations. For the inverse problem, one is interested in determining the conductivity, electric permittivity and/or the magnetic permeability. Zonghou Xiong and Andreas Kirsch consider the geophysical induction problem where the space-dependent conductivity of a buried region in the earth has to be determined from measurements on the surface of the earth. Their method is based on the vector-valued version of the Lipmann–Schwinger equation. This equation is used as the regularizing term for the determination of the current from a volume integral equation of the first kind. In the paper by Erkki Somersalo, David Isaacson and Margaret Cheney, the field equations are linearized and the classical approach of Calderón is used to determine the perturbations in the magnetic permeability, electric permittivity and conductivity from measurements of the fields on the surface of the body. A detailed proof of an estimate of the residual term with respect to the linearization is given. Finally, Sailing He and Staffan Ström study the one-dimensional, time-dependent electromagnetic inverse problem for a point source above an inhomogeneous dissipative slab. Two inverse algorithms based on invariant imbedding equations are used to reconstruct both the permittivity and the conductivity. Numerical results are presented for both exact and noisy data.

References

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Guest Editor